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A STUDY OF THE PLANE UNRESTRICTED THREE-BODY PROBLEM†

V. N. TKHAI

Moscow

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The general (unrestricted) three-body problem is investigated in the case when the force of mutual attraction between the bodies is proportional to the nth power of their distance, where n is an arbitrary real number. A new description is given of the plane problem, based on the introduction of the following Lagrange variables: r—the square root of half the polar moment of inertia, ψ—the angle between the two sides of the triangle, and y—the natural logarithm of the quotient of those two sides. The first variable characterizes the size of the triangle, and the other two, its configuration. Routh's equations are derived, in which the variable r is 'almost separated' from y and ψ; the system of equations is reversible. In the special case of the restricted problem, i.e. when the mass of one of the bodies tends to zero, the variables are completely separable, so that the problem describes only the change in the configuration of the triangle.

It is shown that the qualitative results, known for Newtonian interaction ($n = -2$), are valid throughout the range $-3 < n < -1$. In particular, for these values of n 'elementary' methods of analysis are used to solve the problems of Hill stability for a pair of bodies, the existence of final motions relating to hyperbolic-elliptic motions is established for $n = -2$, and a local analysis is carried out of the neighbourhoods of the classical liberation points.

Local analysis showed that in the neighbourhood of collinear points two families of Lyapunov periodic motions exist, into which the family of two-dimensional "whiskered" tori degenerates. In the linear approximation, the problems of the stability of triangular points in the restricted and unrestricted formulations are equivalent to one another. Hence the triangular elliptical solutions of the unrestricted problem are stable throughout the domain constructed by Danby for the restricted problem. Allowance for the small non-zero mass of one of the bodies may make the other two bodies leave the unperturbed circular orbit; there is no such effect in the restricted problem. Copyright © 1996 Elsevier Science Ltd.

1. THE VARIABLES OF THE PROBLEM. ROUTH'S FUNCTION

The unrestricted three-body problem, i.e. the problem of the motion of a mechanical system consisting of three point masses $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_2$ of masses M_0, M_1 and M_2 which attract one another according to Newton's law, has a 300-year-history and was first considered by Newton himself. We shall study the three-body problem for an interaction force in which the force F_{ij} with which the points $\mathbf{P}_i$ and $\mathbf{P}_j$ attract one another is proportional to an arbitrary power n ($n \neq -1$) of their distance r_{ij}

$$F_{ij} = fM_iM_jr_{ij}^n \quad (i,j = 0,1,2;\ i \neq j) \tag{1.1}$$

where f is some constant. This formulation of the problem goes back to Laplace [1], Routh [2] and Lyapunov [3] and has applications not only in celestial mechanics but also in stellar dynamics. The value $n = -2$ corresponds to Newtonian attraction.

The points $\mathbf{P}_1$ and $\mathbf{P}_2$ are assumed to be moving relative to a system of coordinates with origin at $\mathbf{P}_0$ (Fig. 1).

Then the absolute velocity of $\mathbf{P}_0$ is

$$\mathbf{v}_0 = -(m_1\mathbf{v}_{1r} + m_2\mathbf{v}_{2r}), \quad m_i = M_i / M \quad (i = 0,1,2), \quad M = M_0 + M_1 + M_2 \tag{1.2}$$

where $\mathbf{v}_{1r}$, $\mathbf{v}_{2r}$ are the relative velocities of $\mathbf{P}_1$ and $\mathbf{P}_2$, respectively. Substituting these values into the expression for the kinetic energy T we obtain

$$2TM^{-1} = \sum_{i=1}^{2} m_i(1-m_i)\mathbf{v}_{ir}^2 - 2m_1m_2\mathbf{v}_{1r}\mathbf{v}_{2r} \tag{1.3}$$

or, in polar coordinates r_i, θ_i ($i = 1, 2$)

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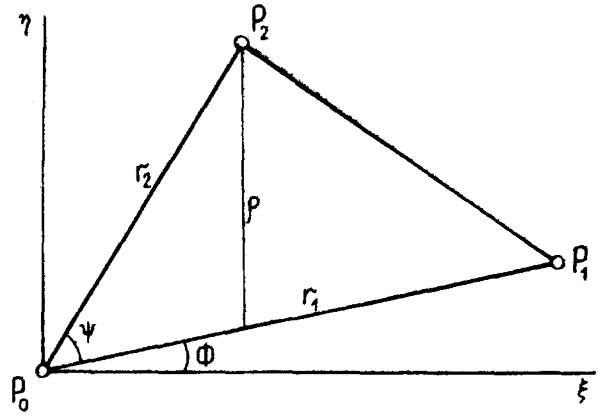

Fig. 1.

$$2TM^{-1} = m_1(1-m_1)(r_1^{\cdot 2} + r_1^2\theta_1^{\cdot 2}) + m_2(1-m_2)(r_2^{\cdot 2} + r_2^2\theta_2^{\cdot 2}) -$$

$$-2m_1m_2\{[r_1^{\cdot}r_2^{\cdot} + r_1r_2\theta_1^{\cdot}\theta_2^{\cdot}]\cos(\theta_2-\theta_1) + (r_1r_2^{\cdot}\theta_1^{\cdot} - r_2r_1^{\cdot}\theta_2^{\cdot})\sin(\theta_2-\theta_1)\}$$

Next, introducing new dimensionless parameters $\mu_{i+j} = m_im_j$ ($i, j = 0, 1, 2; i \neq j$) and taking into account that $\theta_2 = \Phi + \psi(\Phi = \theta_1)$, we have

$$2TM^{-1} = (\mu_1+\mu_3)(r_1^{\cdot 2} + r_1^2\Phi^{\cdot 2}) + (\mu_2+\mu_3)[r_2^{\cdot 2} + r_2^2(\psi^{\cdot} + \Phi^{\cdot})^2] -$$

$$-2\mu_3\{[r_1^{\cdot}r_2^{\cdot} + r_1r_2\Phi^{\cdot}(\psi^{\cdot}+\Phi^{\cdot})]\cos\psi + [r_1r_2^{\cdot}\Phi^{\cdot} - r_2r_1^{\cdot}(\psi^{\cdot}+\Phi^{\cdot})]\sin\psi\}$$

It is readily seen that the variable Φ is cyclic, corresponding to the area integral. Carrying out the procedure of ignoring a cyclic coordinate and forming the Routh function of the problem, we obtain

$$\beta = M^{-1}\partial T/\partial\Phi^{\cdot} = [\mu_1 + \mu_2e^{2y} + \mu_3(1+e^{2y} - 2e^y\cos\psi)]r_1^2\Phi^{\cdot} +$$

$$+[(\mu_2+\mu_3)e^{2y} - \mu_3e^y\cos\psi]r_1^2\psi^{\cdot} - \mu_3e^yr_1^2\sin\psi\, y^{\cdot}$$

$$2R = [\mu_1+\mu_3+(\mu_2+\mu_3)e^{2y} - 2\mu_3e^y]r_1^{\cdot 2} + (\mu_2+\mu_3)e^{2y}[r_1^2(y^{\cdot 2}+\psi^{\cdot 2}) + 2r_1r_1^{\cdot}y^{\cdot}] -$$

$$-2\mu_3e^y[r_1r_1^{\cdot}y^{\cdot}\cos\psi - r_1r_1^{\cdot}\psi^{\cdot}\sin\psi] - (\mu_1r_1^2 + \mu_2r_2^2 + \mu_3r_3^2)\Phi^{\cdot 2} + U_*$$

$$U_* = \frac{U}{M}, \quad U = -\frac{f}{n+1}\{M_0M_1r_1^{n+1} + M_0M_2r_2^{n+1} + M_1M_2r_3^{n+1}\}, \quad y = \ln\frac{r_2}{r_1}$$

where r_3 is the distance between the points $\mathbf{P}_1$ and $\mathbf{P}_2$, and U is the force function of the problem.
To continue, we replace the distance r_1 with a new variable r

$$2r^2 = \mu_1r_1^2 + \mu_2r_2^2 + \mu_3r_3^2 = \frac{M_0M_1r_1^2 + M_0M_1r_2^2 + M_1M_2r_3^2}{(M_0+M_1+M_2)^2}$$

which has the meaning of the square root of half the polar moment of inertia, divided by the mass of the entire system. This choice is dictated by the following consideration. The Lagrange–Jacobi relation [4, 5]—the differential equation for the polar moment of inertia, is of fundamental importance in celestial mechanics. This equation is a differential expression of the law of conservation of total mechanical energy and has been the subject of many studies [6–11]. When r, y and ψ are taken as independent variables, one of the equations of motion is in fact the Lagrange–Jacobi equation, while the other equations close the system of differential equations for the motion of the three-body system.

Omitting the very laborious intervening algebra, we can write the final expression for the Routh function

$$R = r'^2 + r^2 F_2 + F_1 - \frac{\beta^2}{4r^2} + r^{n+1} F_0 \quad (n \neq -1) \tag{1.4}$$

$$F_2 = \frac{\mu}{S^2}(y'^2 + \psi'^2) \quad (\mu = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3)$$

$$F_1 = -\frac{\beta}{S}[\mu_3(\psi' \cos\psi + y' \sin\psi) - (\mu_2 + \mu_3)e^y \psi']$$

$$F_0 = -\frac{fM}{n+1}\left(\frac{2}{S}\right)^{(n+1)/2} \{\mu_1 e^{-(n+1)y/2} + \mu_2 e^{(n+1)y/2} + \mu_3(e^y + e^{-y} - 2\cos\psi)^{(n+1)/2}\}$$

$$S = \mu_1 e^{-y} + \mu_2 e^y + \mu_3(e^y + e^{-y} - 2\cos\psi)$$

Now, knowing the Routh function, we can derive the Hamiltonian of the problem

$$H = \frac{p_r^2}{4} + \frac{S^2}{4\mu r^2}[(p_\psi - b_\psi)^2 + (p_y - b_y)^2] + \frac{\beta^2}{4r^2} - r^{n+1} F_0 \tag{1.5}$$

$$b_\psi = -\frac{\beta}{S}[\mu_3 \cos\psi - (\mu_2 + \mu_3)e^y], \quad b_y = -\frac{\beta}{S}\mu_3 \sin\psi$$

where p_r, p_ψ, p_y are the momenta corresponding to the variables r, ψ, y, respectively.

The distinctive features of this description of the problem are as follows: The new parameters of the problem are the dimensionless products μ_{ij} of the masses of the bodies $\mathbf{P}_i$ and $\mathbf{P}_j$, and these parameters reflect the interaction of the bodies. The equation for the variable r

$$2r'' = 2rF_2 + \frac{\beta^2}{2r^3} + (n+1)r^n F_0 \tag{1.6}$$

is in fact the Lagrange–Jacobi equation. Finally, the problem is described by the Routh or Hamilton equations, which are also reversible. Thus, in the case of the Routh equations, the fixed set is the hyperplane $\mathbf{M} = \{r, y, \psi, r', y', \psi'{:}\ \psi = 0 (\mathrm{mod}\ \pi), r' = 0, y' = 0\}$, and for the Hamilton equations it is $\mathbf{M}_1 = \{r, y, \psi, p_r, p_y, p_\psi{:}\ \psi = 0 (\mathrm{mod}\ \pi), p_r = 0, p_y = 0\}$.

It is also noteworthy that the part of the Routh function (1.4) that is quadratic in the velocities is already reduced to "isometric" coordinates [12, p. 537].

2. SOME COROLLARIES OF THE ENERGY INTEGRAL

The system of equations of motion with the Routh function (1.4) has an energy integral

$$r'^2 + r^2 F_2 + \frac{\beta^2}{4r^2} - r^{n+1} F_0 = h \quad (h = \mathrm{const}) \tag{2.1}$$

It follows from the form of the function $F_0(y, \psi)$ that $F_0(y, \psi)$ takes positive (negative) values for $n < -1$ ($n > -1$), and it is unbounded if $n < -1$ and tends to $+\infty$ as $y \to \pm\infty$. If $n > -1$, then $F_0(y, \psi)$ is bounded.

Consider the function

$$g(p, q) = P^{-(n+1)/2} Q$$

$$P = \mu_1 + \mu_2 p + \mu_3 q, \quad Q = \mu_1 + \mu_2 p^{(n+1)/2} + \mu_3 q^{(n+1)/2}$$

which is identical with F_0 apart from a constant factor if one puts

$$p = e^{2y}, \quad q = 1 + e^{2y} - 2e^{y}\cos\psi$$

In the domain $p > 0, q \geqslant 0$ this function has a unique stationary point $p = q = 1$, since $P(p, q) > 0$. Thus the function F_0, apart from the stationary points $y = 0$, $\psi = \pm\pi/3$ (the Lagrangian triangular solutions), can have only stationary points such that

$$\partial q/\partial\psi = 2e^{y}\sin\psi = 0$$

that is, collinear solutions.

Let us determine the nature of the extremum of $F(p, q)$ at the point $p = q = 1$. Calculations show that if $n^2 > 1$, the function $g(p, q)$ has a global minimum at $p = q = 1$, equal to $g^\circ = \nu^{(1-n)/2}$, $\nu = \mu_1 + \mu_2 + \mu_3$; the degree sign means that the functions are evaluated at the stationary point $p = q = 1$. Hence it follows that the values of F_0 at the collinear stationary points, when $n < 1$ ($n > 1$), are greater (smaller) than their values at the Lagrangian triangular points.

Let us proceed now to the conclusions.

Theorem 1. If $n > 1$, the Lagrangian triangular solutions of the unrestricted three-body problem are secularly stable for any values of the masses.

Remark. The existence of the triangular solutions for a Newtonian law was first established by Lagrange [13]. Laplace [1] extended the result to the case of arbitrary n.

The proof of Theorem 1 follows from the fact that $g(p, q)$ is positive definite in the neighbourhood of the point $p = q = 1$, and from the energy integral (2.1). Routh's theorem [14] with Lyapunov's supplement [15] are used here.

The Lagrangian triangular solutions are found from the condition for the function R_0 to be stationary. For this solution

$$R_0^\circ = \frac{n-1}{n+1}\frac{\beta^2}{4r_0^2}, \quad r_0 = r^\circ = \left[\frac{\beta^2}{4fM}\left(\frac{\nu}{2}\right)^{(n-1)/2}\right]^{1/(n+3)}$$

In the neighbourhood of a stationary point

$$R_0 - R_0^\circ = -\frac{\beta^2}{2r_0^4}(n+3)\delta r^2 -$$
$$-\frac{\beta^2}{2r_0^2}\frac{(n-1)\mu_3}{\nu}\left\{\left[1 - \frac{3}{4}(\mu_1 + \mu_2)\right]\delta\psi^2 + 2\sqrt{3}(\mu_1 - \mu_2)y\delta\psi + \frac{3}{4}(\mu_1 + \mu_2)y^2\right\} + \ldots,$$

$$\delta r = r - r^\circ, \quad \delta\psi = \psi - \pi/3$$

Hence it is clear that when $n < -3$ the number of negative Poincaré coefficients is three, and, by a well-known theorem due to Kelvin–Chetayev [16], the Lagrangian triangular solutions are unstable. When $-3 < n < 1$ the degree of instability is two, and gyroscopic stabilization is possible; conditions to that end were determined by Gascheau [17], Routh [2] and Zhukovskii [18].

Theorem 1 is a corollary of the fact that the function $-R_0(r, y, \psi)$ attains a local minimum at the point $r = r^\circ, y = 0, \psi = \pi/3$ when $n > 1$. The fact that the minimum at $p = q = 1$ of the function $g(p, q)$ is global, together with the form of the function R_0, imply that $r = r^\circ, y = 0$ is a global minimum point of the function $-R_0$ when $n > 1$. This implies that the bodies cannot collide if $h < h^*$ (this upper bound will be determined below). The fact that a triple collision cannot occur when $n > -1$ follows from the energy integral (2.1). In a double collision $y \to \pm\infty$ or $y \to \infty$, $\psi \to 0$. In this cases g cannot exceed one of the numbers

$$(\mu_2 + \mu_3)^{(n-1)/2}, \quad (\mu_1 + \mu_3)^{(n-1)/2}, \quad (\mu_1 + \mu_2)^{(n-1)/2}$$

Therefore, if

$$-F_0 \leqslant \left(h - \frac{\beta^2}{4r^2}\right) r^{-(n+1)}$$

we obtain $-F_0 < b/(n+1)$, and a double collision is impossible. The function $\beta^2/(4r^2) + br^{n+1}$, however, is always greater than

$$h^* = \frac{1}{2}\frac{n+3}{n+1} b \left(\frac{\beta^2}{2b}\right)^{(n+1)/(n+3)}$$

Theorem 2. When $n > -1$ the motion in the unrestricted three-body problem is stable in Lagrange's sense and triple collisions are impossible. When $n > 1$ motion with energy constant $h < h^*$ occurs without double and triple collisions.

Now let $-3 < n < -1$ and suppose that $h < 0$. It follows from the energy integral that

$$F_0(y, \psi) \geqslant G(r) = \frac{\beta^2}{4} r^{-(n+1)} - hr^{-(n+3)} \tag{2.2}$$

In the range of n values under consideration, the function $G(r)$ reaches a minimum value

$$\chi = -\frac{2h}{n+3}\left[\frac{\beta^2}{4h}\frac{n+3}{n+1}\right]^{-(n+1)/2} \quad \text{for} \quad r^2 = \frac{\beta^2}{2h}\frac{n+1}{n+3}$$

where $\chi^2 = -h\beta^2$ in the case of Newtonian interaction.

The inequality

$$F_0(y, \psi) \geqslant \chi \tag{2.3}$$

is known in celestial mechanics [19] in the case $n = -2$ as Golubev's inequality; it is used to solve the problem of Hill stability in the unrestricted three-body problem.

The function F_0 has a global minimum at the points $L_{4,5}(0, \pm\pi/3)$, and also three collinear saddle-points L_j ($j = 1, 2, 3$). The proof of the existence of just the three collinear points, the nature of the extremum of F_0 at these points and the comparison of the values of F_0 at these points is extremely cumbersome and is omitted here. We will merely point out that it is based on investigating the properties of the function $S(\xi) = \partial g/\partial\xi$ ($\xi = e^y$, $\sin\psi = 0$), and on determining the sign of $\partial^2 g/\partial\psi^2$ at points where $S(\xi) = 0$. One must take into consideration that the mixed derivative of $q(\xi, \psi)$ vanishes at the collinear points.

In the general case $M_0 > M_1 > M_2$, we have

$$F_0(\mathbf{L}_2) > F_0(\mathbf{L}_3) > F_0(\mathbf{L}_1) > F_0(\mathbf{L}_4) = F_0(\mathbf{L}_5)$$

and the level curves of F_0 are as shown in Fig. 2. We choose χ as the parameter of the problem. Then for $\chi > F_0(\mathbf{L}_2)$ the motion will occur in the hatched domains in the figure, and one obtains Hill stability of two bodies, $\mathbf{P}_0$ and $\mathbf{P}_1$ (domain 1), $\mathbf{P}_0$ and $\mathbf{P}_2$ (domain 3), or $\mathbf{P}_1$ and $\mathbf{P}_2$ (domain 2). When $F_0(\mathbf{L}_3) < \chi < F_0(\mathbf{L}_2)$ the pair of bodies $\mathbf{P}_0$ and $\mathbf{P}_1$ is Hill stable, but when $\chi < F_0(\mathbf{L}_3)$ Hill stability can no longer be guaranteed for any pair of bodies.

Now, considering the special cases in which some of the bodies $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$ have the same mass, one can extend to the range $-3 < n < -1$ all conclusions obtained in [19–22] for the case $n = -2$. The qualitative aspect of the problem is quite clear if one bears [19–22] in mind. The actual calculations are omitted.

One more problem needs attention. It follows from (2.2) that in the range $-3 < n < -1$ one has $F_0 \to \infty$ as $r \to \infty$ or $r \to 0$. In that case one body ($\mathbf{P}_2$) will always remain relatively distant from the pair $\mathbf{P}_0$ and $\mathbf{P}_1$. This immediately implies, for example, Sludskii's theorem (see, e.g. [4]) that triple collisions are impossible when $\beta^2 \neq 0$.

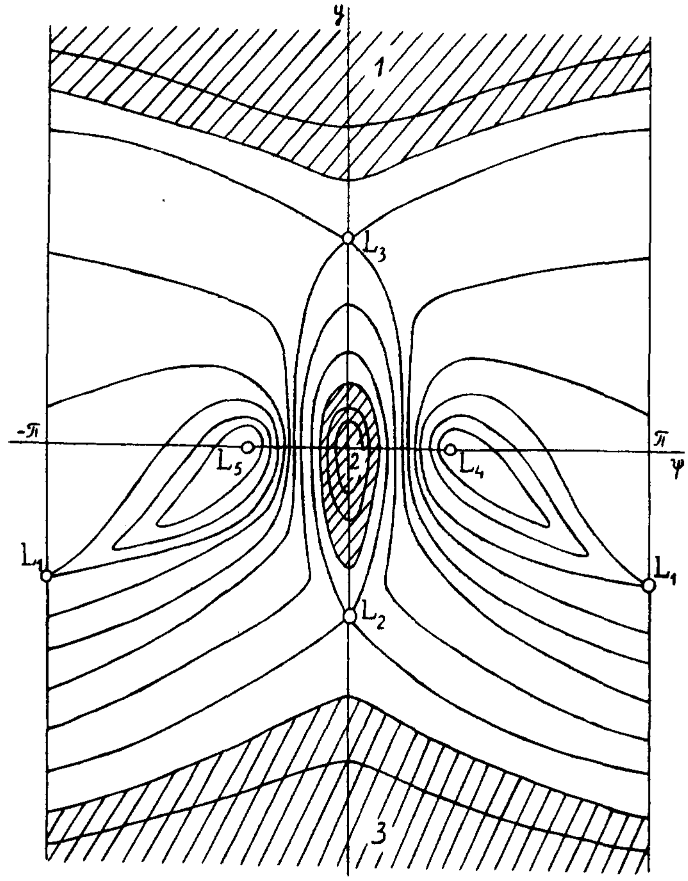

Fig. 2.

3. EXTENSION OF A THEOREM OF BIRKHOFF

A comparison of the differential equation (1.6) and the expression (2.1) for the energy integral immediately leads to the important differential equality

$$dF_* / dr = (n+3)r^{-n}F_2, \quad F_* = F_0 - r^{1-n}F_2 \tag{3.1}$$

For every r, the function $-F_*$ may be treated as the energy determined solely by the configuration of the triangle $\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2$ but not by its size. It follows from (3.1) that when $n > -3$ the function increases and decreases together with r, while when $n = -3$ the system has an additional first integral $F_* =$ const. Thus, when $n > -3$—this includes Newtonian attraction—the law of conservation of energy is satisfied in such a way that, as the polar moment of inertia increases, the energy determined only by the configuration of the triangle decreases if the configuration varies at the same time; an energy transfer occurs. We also note that the inequality $dF_*/dr \geqslant 0$ for Newtonian interaction was first derived by Sundman [8]; it was fundamental to his verification that the series he constructed do indeed converge.

Equation (1.6) may also be formulated for the polar moment of inertia $J = 2r^2$. If one then takes the energy integral (2.1) into account, this gives

$$J^{\cdot\cdot} = 2(n+3)U_* + 2h \tag{3.2}$$

Hence it follows that if $n \leqslant -3$, any motion with non-positive energy ($h \leqslant 0$) includes a triple collision. In that case there are no motions for which $r \to \infty$. Indeed, letting $r \to \infty$ in (2.1), we get $F_* \to \infty$, which is impossible by (3.1). When $n > -1$ the energy, as follows from the integral (2.1), is always positive; this case was considered in Section 2.

Birkhoff's result [6] holds for a more general interaction (1.1), provided that the exponent n lies in the range $-3 < n < -1$. The proof (see [23]) follows the arguments used in [6] for the case $n = -2$

almost word for word; it is based on the inequality

$$\rho > \sqrt{-\frac{fM}{(n+1)}\left(\frac{\rho}{2}\right)^{(n+1)/2}} \tag{3.3}$$

first obtained by Birkhoff for the case $n = -2$; here ρ is the distance from the body $\mathbf{P}_2$ to the mass centre of $\mathbf{P}_0$ and $\mathbf{P}_1$ (Fig. 1).

It follows from the energy integral (2.1) that $r^{n+1}F_0 + h \geqslant 0$. Hence r_1, the least of the distances r_1, r_2 and r_3, does not exceed a certain limit

$$r_1 \leqslant \left[\frac{fM\nu}{(n+1)h}\right]^{-1/(n+1)} = r_*$$

If $\rho > 2r_*$, then $r_2 \geqslant \rho - r_1 > \rho/2$, $r_3 \geqslant \rho - r_1 > \rho/2$ and the same distance is the smallest (Fig. 1). If at the same time condition (3.3) holds and the inequality $\rho > 2r_*$ is true, both these conditions will continue to hold at all times, proving that ρ will increase without limit as $t \to +\infty$. If $\rho \to \infty$, then also $r \to +\infty$, because

$$2r^2 = \varkappa r_1^2 + (\mu_2 + \mu_3)\rho^2, \quad \varkappa = \frac{\mu_1}{m_0 + m_1}, \quad \lim_{\rho\to+\infty} \frac{2r^2}{\rho} = \mu_2 + \mu_3 \neq 0$$

We will now show that if $r \leqslant d$, where d is some sufficiently small number, we have $r \to +\infty$ as $t \to +\infty$. First, it follows from the energy interval that $\dot{r}_1^2 \leqslant 2U$. Thus

$$\varkappa r_1^2 r_1^{\cdot 2} \leqslant -\frac{2fM\nu}{n+1} r_1^{n+3} \leqslant -\frac{2fM\nu}{n+1} r_*^{n+3} \quad (n+1 < 0)$$

In addition, we have

$$2rr^{\cdot} = \varkappa r_1 r_1^{\cdot} + (\mu_2 + \mu_3)\rho\rho^{\cdot}$$

Therefore, if

$$2rr^{\cdot} > \varkappa\sqrt{-\frac{2fM\nu}{(n+1)\varkappa}} r_*^{(n+3)/2} + (\mu_2 + \mu_3)\sqrt{-\frac{fM}{(n+1)2^n}} \rho^{(n+3)/2} \tag{3.4}$$

then inequality (3.3) is also true. The right-hand side of inequality (3.4) may be written in a form symmetric with respect to μ_1, μ_2, μ_3 depending only on r. We have

$$2r^2 \geqslant (\mu_2 + \mu_3)\rho^2, \quad \rho^{(n+3)/2} \leqslant (2r^2)^{(n+3)/4}(\mu_2 + \mu_3)^{-(n+3)/4}, \quad \mu_2 + \mu_3 \leqslant \nu$$

Therefore, inequality (3.4) will certainly be true if

$$2rr^{\cdot} \geqslant \sqrt{-\frac{2fM\nu^2}{n+1}} r_*^{(n+3)/2} + \sqrt{-\frac{fM}{n+1}} 2^{(3-n)/4} \nu^{(1-n)/4} r^{(n+3)/2}$$

Now, for $r_1 < r_*$ we obtain

$$(\mu_2 + \mu_3)\rho^2 = 2r^2 - \varkappa r_1^2 > 2r^2 - \varkappa r_*^2$$

Therefore, $\rho > 2r_*$ if

$$2r^2 \geqslant 5\nu r_*^2 > [\varkappa + 4(\mu_2 + \mu_3)] r_*^2$$

Thus, if the system reaches the domain

$$r \geqslant \sqrt{\frac{5\nu}{2}} r_*, \quad rr^{\cdot} \geqslant \sqrt{-h}\left[\sqrt{\frac{\nu}{2}} + \left(\frac{r}{r_*}\right)^{(n+3)/2} (2\nu)^{-(n+1)/4}\right] \tag{3.5}$$

at some time, it will remain there, and moreover $r \to +\infty$ as $t \to +\infty$.

We now rewrite the energy integral in the form

$$F_* = \frac{\beta^2}{4} r^{-(n+3)} + (r^{\cdot 2} - h) r^{-(n+1)} \tag{3.6}$$

The least value of F_* on the boundary of the domain (3.5) is defined by the expression

$$A = \left\{\frac{\beta^2}{4} - \frac{5\nu h}{2}\left(\left[\sqrt{\frac{\nu}{2}} + (2\nu)^{-(n+1)/4}\right]^2 + 1\right) r_*^2\right\}\left(\sqrt{\frac{5\nu}{2}} r_*\right)^{-(n+3)} \tag{3.7}$$

It follows from (3.1) that if at some time the system was in one of the domains

$$r \leqslant d \tag{3.8}$$

$$d \leqslant r \leqslant r_{**}, \quad r^{\cdot} \geqslant 0, \quad F_* \geqslant A \tag{3.9}$$

then after some time it will necessarily reach the domain (3.5) and $r \to +\infty$ as $t \to +\infty$. By (3.5) and (3.6), the values of d and r_{**} are defined by the following relations

$$\frac{\beta^2}{4} d^{-(n+3)} - h d^{-(n+1)} = A$$

$$\frac{\beta^2}{4} r_{**}^{-(n+3)} + (r_{**}^{\cdot 2} - h) r_{**}^{-(n+1)} = A, \quad r_{**} r_{**}^{\cdot} = \sqrt{-h}\left[\sqrt{\frac{\nu}{2}} + \left(\frac{r_{**}}{r_*}\right)^{(n+3)/2} (2\nu)^{-(n+1)/4}\right]$$

where d is the least and r_{**} the greatest root of the appropriate equations.

Theorem 3. If the system reaches one of the domains (3.5), (3.8) or (3.9) at some time, then as the motion progresses one of the bodies will go off to infinity and the other two form a Hill-stable pair of bodies.

Note that, because of the existence of the domains (3.5) and (3.9), one of the bodies can go off to infinity not only from a position close to triple collision. This has been verified numerically [9, 10].

Condition (3.8) is sufficient for one of the bodies to go to infinity. This is also possible in the case when the system of bodies belongs to the fixed set **M**. In such motions, the body $\mathbf{P}_2$ approaches $\mathbf{P}_0$ and $\mathbf{P}_1$ from infinity, forming at a certain time a rectilinear configuration with them; it then leaves the two-body system $\mathbf{P}_0$ and $\mathbf{P}_1$ and goes off to infinity.

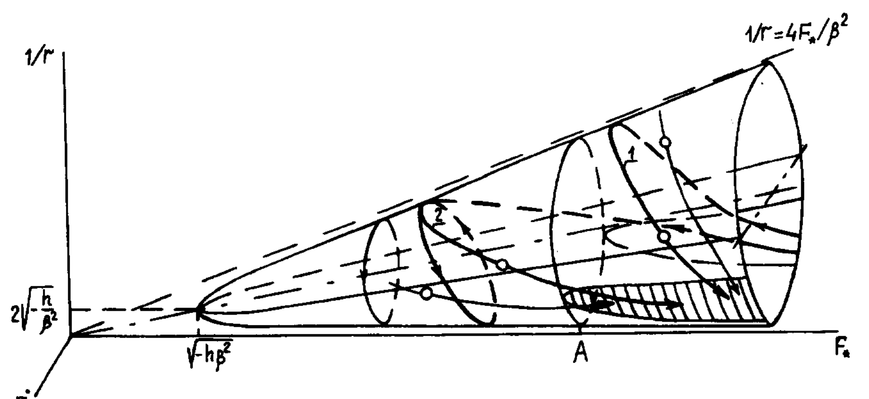

Fig. 3.

A descriptive geometrical interpretation of the results may be obtained in three-dimensional space ($\dot{r}$, $1/r$, F_*). In that space formula (3.6) is the equation of the surface of the energy integral for every (β, h) (Fig. 3). Formula (3.1) means that over the visible part of the surface ($\dot{r} > 0$) the motion takes place to the right of the plane $F_* = A$. The intersection of this plane with the curve $\dot{r} = 0$ defines d, and the domain (3.5) is hatched. Clearly, the system may reach (3.5), generally speaking, from any point of the surface as represented. The Birkhoff curve is numbered 1 and the Szebehely curve is numbered 2.

Steady-state solutions are represented by the extreme left point of the surface, and periodic solutions with fixed triangle configuration are represented by the curves cut from the surface by planes $F_* =$ const.

Note that Fig. 3 makes it very easy to derive the results of [6–11] pertaining to the behaviour of the variable r between two neighbouring extremal values.

4. EQUATIONS OF MOTION

The equations governing the variation of the configuration of the triangle are as follows:

$$\frac{d}{dt}\left(r^2\frac{\partial F_2}{\partial \dot{\xi}}+\frac{\partial F_1}{\partial \dot{\xi}}\right)=r^2\frac{\partial F_2}{\partial \xi}+\frac{\partial F_1}{\partial \xi}+r^{n+1}\frac{\partial F_0}{\partial \xi}$$

where ξ denotes the variables y and ψ. We introduce a new variable angle θ defined by

$$2r^2 d\theta/dt = \beta$$

Then

$$\frac{d}{d\theta}\left[\frac{\mu}{S^2}\psi'-\frac{\mu_3\cos\psi-(\mu_2+\mu_3)e^y}{S}\right]=\frac{\mu}{2}\frac{\partial F_2^*}{\partial\psi}+\frac{\partial F_1^*}{\partial\psi}+\frac{r^{n+3}}{\beta^2}\frac{\partial F_0^*}{\partial\psi}$$

$$\frac{d}{d\theta}\left[\frac{\mu}{S^2}y'-\frac{\mu_3\sin\psi}{S}\right]=\frac{\mu}{2}\frac{\partial F_2^*}{\partial y}+\frac{\partial F_1^*}{\partial y}+\frac{r^{n+1}}{\beta^2}\frac{\partial F_0^*}{\partial y}$$

where now

$$F_2^* = \frac{1}{2}(\psi'^2+y'^2)$$

$$F_1^* = -\frac{1}{S}[\mu_3(\psi'\cos\psi+y'\sin\psi)-(\mu_2+\mu_3)e^y\psi']$$

$$F_0^* = -\frac{fM}{n+1}[\mu_1 e^{-(n+1)y/2}+\mu_2 e^{(n+1)y/2}+\mu_3(e^y+e^{-y}-2\cos\psi)^{(n+1)/2}]S^{-(n+1)/2}$$

The prime denotes differentiation with respect to θ, and r is replaced by $\sqrt{2}r$.

After evaluating the derivatives and performing the necessary reduction, we obtain the system

$$\frac{d}{d\theta}\left(\frac{\psi'}{S^2}\right)=-\frac{2y'}{S^2}+\frac{1}{2}\frac{\partial F_2^*}{\partial\psi}+\frac{r^{n+3}}{\mu\beta^2}\frac{\partial F_0^*}{\partial\psi}$$
$$\frac{d}{d\theta}\left(\frac{y'}{S^2}\right)=\frac{2\psi'}{S^2}+\frac{1}{2}\frac{\partial F_2^*}{\partial y}+\frac{r^{n+3}}{\mu\beta^2}\frac{\partial F_0^*}{\partial y} \quad (4.1)$$

Finally, adding the equation for r to system (4.1)

$$r^{\cdot\cdot} = rF_2 + \frac{\beta^2}{r^3} + (n+1)r^n F_0^* \tag{4.2}$$

we obtain a compact system of differential equations describing the plane unrestricted three-body problem.

Let us evaluate the derivatives of the function F_0^*

$$\frac{\partial F_0^*}{\partial \psi} = -fM\mu_3 P^{-(n+3)/2}(Pq^{(n-1)/2} - Q)e^y \sin\psi$$

$$\frac{\partial F_0^*}{\partial y} = -fMP^{-(n+3)/2}[\mu_2(Pp^{(n-1)/2} - Q)e^y + \mu_3(Pq^{(n-1)/2} - Q)(e^y - \cos\psi)]e^y$$

As $m_2 \to 0$ we have $\mu_2/\mu \to 1/m_1$, $\mu_3/\mu \to 1/m_0$, and in that case Eqs (4.1) and (4.2) yield a version of the problem in which the mass of $\mathbf{P}_2$ is negligibly small compared to the masses of $\mathbf{P}_0$ and $\mathbf{P}_1$. The equations of motion of the limiting problem are

$$\frac{d}{d\theta}(e^{2y}\psi') = -2e^{2y}y' - \frac{fMr_1^{n+3}}{\beta_*^2} m_1[(1+e^{2y} - 2e^y\cos\psi)^{(n-1)/2} - 1](e^y - \cos\psi)e^y$$

$$\begin{aligned}\frac{d}{d\theta}(e^{2y}y') &= 2e^{2y}\psi' + e^{2y}(\psi'^2 + y'^2) - \frac{fMr_1^{n+3}}{\beta_*^2}\times \\ &\times\{m_0(e^{ny} - e^y)e^y + m_1[(1+e^{2y} - 2e^y\cos\psi)^{(n-1)/2} - 1](e^y - \cos\psi)e^y\}\end{aligned} \tag{4.3}$$

$$r_1^{\cdot\cdot} = \frac{\beta_*^2}{r_1^2} - fMr_1^n, \quad \beta_* = \beta/\mu_1$$

The body $\mathbf{P}_2$ of zero mass obviously has no influence on the finite-mass bodies $\mathbf{P}_0$ and $\mathbf{P}_1$. Consequently, in the limit as $m_2 \to 0$ Eqs (4.1) and (4.2) of the unrestricted three-body problem imply Eqs (4.3) of the restricted three-body problem. The latter is described by Lagrange's equations with Lagrangian

$$L = L_2 + L_1 + L_0$$

$$L_2 = \frac{1}{2}e^{2y}(\psi'^2 + y'^2), \quad L_1 = e^{2y}\psi'$$

$$L_0 = -\frac{fMr_1^{n+3}}{(n+1)\beta_*^2}\left\{m_0\left(p^{(n+1)/2} - \frac{n+1}{2}p\right) + m_1\left(q^{(n+1)/2} - \frac{n+1}{2}q\right)\right\}$$

$$p = e^{2y}, \quad q = 1 + e^{2y} - 2e^y\cos\psi$$

and in the case of the circular restricted problem (r_1 = const) we have an energy integral

$$L_2 - L_0 = \text{const}$$

Thus, the restricted problem describes only the variation in the configuration of the triangle $\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2$, not in its size.

In the restricted three-body problem, one studies the motion of a point of zero mass in the field of attraction of two bodies of finite mass, on the assumption that the zero-mass body $\mathbf{P}_2$ has no influence on the motion of the bodies $\mathbf{P}_0$ and $\mathbf{P}_1$.

However, a slight change in the motion of $\mathbf{P}_0$ and $\mathbf{P}_1$ may have a major influence on the motion of

$\mathbf{P}_2$. Thus, if the originally circular orbit of $\mathbf{P}_0$ and $\mathbf{P}_1$ is replaced by a neighbouring elliptical orbit, parametric resonance and instability of the triangular libration points may occur [24]. If the body is of relatively small—but not zero—mass, then $\mathbf{P}_2$ will affect the motion of $\mathbf{P}_0$ and $\mathbf{P}_1$, and this may imply qualitatively new conclusions [25] compared to the restricted formulation of the problem. Finally, the fact that system (4.1) is "almost separable" from Eq. (4.2) should lead to new qualitative results in the three-body problem, at least for small m_2.

5. THE NEIGHBOURHOOD OF THE LIBRATION POINTS

The stationary points of the new function $F_0^*(\psi, y)$ represent the motion of the system $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$ with a fixed triangle configuration. In that situation the body $\mathbf{P}_2$ is situated at one of the classical libration points $\mathbf{L}_j$, and the variable r obeys the equation

$$r^{\cdot\cdot} = \frac{\beta^2}{r^3} + (n+1)r^n F_0^*(\mathbf{L}_j) \tag{5.1}$$

where the function F_0^* is evaluated at $\mathbf{L}_j$. This equation has a periodic solution—in particular, a constant solution—for any $n > -3$.

Let us form the variational equations in the neighbourhood of the constant solution corresponding to some point $\mathbf{L}_j$. It turns out that the equation for r has the same form as if it had been set up for Eq. (5.1)

$$\delta r^{\cdot\cdot} = -\frac{\beta^2}{r^4}(n+3)\delta r \tag{5.2}$$

Hence it is clear that when $n > -3$ constant solutions correspond to a pair of pure imaginary roots of the characteristic equation. Irrespective of the dependence on the values of the other roots, system (4.1), (4.2) has a Lyapunov family of local periodic motions in the neighbourhood of the periodic solution under consideration. This is in fact clear, since Eq. (5.1) has not only a constant solution, but also a neighbouring periodic solution. In addition, there is a Lyapunov family in the neighbourhood of any periodic solution with fixed triangle configuration, and this family may be extended globally until the constant energy integral vanishes.

Let us now consider constant collinear libration points. These points are saddle points of F_0^* and so the variational equations for the variables y, ψ, which are easily derived from (4.1), have a pair of pure imaginary roots and a pair of roots with non-zero real parts that have opposite signs. The pair of pure imaginary roots represents the second family of Lyapunov periodic motions surrounding the libration points (Fig. 4) and symmetric about the fixed set $\mathbf{M}$.

The two Lyapunov families of periodic motions are obtained when two-dimensional tori degenerate over suitable manifolds. These tori are "whiskered"; to a pair of roots with non-zero real parts there correspond two families of motions that are asymptotic to the tori as $t \to +\infty$ and $t \to -\infty$.

In the neighbourhood of the triangular libration points, Eqs (4.1) become

$$\begin{aligned} x'' + 2y' &= u\left[\frac{3}{4}(m_1 + m_2)x + \frac{\sqrt{3}}{4}(m_1 - m_2)y\right] + \ldots \\ y'' - 2x' &= u\left\{\frac{\sqrt{3}}{4}(m_1 - m_2)x + \left[1 - \frac{3}{4}(m_1 + m_2)\right]y\right\} + \ldots \end{aligned} \tag{5.3}$$

where

$$x = \psi - \pi/3, \quad u = fM(1-n)\rho^{n+3}/\beta_*^2, \quad \beta_* = \beta/\nu$$

and the relationship between ρ and θ is given by the equation

$$\rho^{\cdot\cdot} = \beta_*^2\rho^{-3} - fM\rho^n, \quad \rho^2\frac{d\theta}{dt} = \beta_*, \quad r = \sqrt{\nu}\rho \tag{5.4}$$

Applying the Lyapunov transformation [3]

$$X = y + sx, \quad Y = x - sy$$

$$s = \left[\frac{3}{4}(m_1 + m_2) - \lambda\right]\left[\frac{\sqrt{3}}{4}(m_1 - m_2)\right]^{-1}, \quad \lambda^2 - \lambda + \frac{3}{4}\nu = 0$$

we can reduce system (5.3) to the form

$$X'' - 2Y' = (1-\lambda)uX + \ldots, \quad Y'' + 2X' = \lambda u Y + \ldots \tag{5.5}$$

and we obtain, in the linear approximation, a linear reversible system. For constant triangular solutions we have

$$fM\rho^{n+3} = \beta_*^2, \quad u = (1-n)$$

and the roots of the characteristic equation are

$$\varkappa^2 = \frac{-(n+3) \pm \sqrt{(n+3)^2 - 3(n-1)^2\nu}}{2}$$

Therefore, the necessary conditions for Routh–Zhukovskii gyroscopic stabilization become

$$\nu < \frac{1}{3}\left(\frac{3+n}{1-n}\right)^2 \tag{5.6}$$

Under these conditions, the neighbourhood of the triangular libration points is filled by three-dimensional tori which may degenerate into two-dimensional tori or into Lyapunov families of periodic motions. If conditions (5.6) are strongly violated, the neighbourhood will consist of periodic triangular solutions (a Lyapunov family corresponding to a constant configuration) and motions asymptotic to them as $t \to +\infty$ and $t \to -\infty$.

Suppose that the constant value h_* of the energy integral corresponds to constant triangular solutions. Then, for small $h - h_*$, periodic triangular solutions close to the constant ones exist. The period of these motions as functions of θ depends on $h - h_*$ and equals 2π only when $h = h_*$.

Let us investigate the stability of the linear system (5.5) in that case. The function $u(\theta)$ is the sum of a constant $(1-n)$ and a function periodic in θ that vanishes at $h = h_*$. Therefore, we have to investigate a quasi-autonomous reversible system—for details, see [26].

Let condition (5.6) be satisfied. Then $\varkappa_j = \pm i k_j$ $(k_j > 0, j = 1, 2)$, where

$$k_{1,2}^2 = \frac{n + 3 \pm \sqrt{(n+3)^2 - 3(n-1)^2\nu}}{2}$$

According to [26], when there is no parametric resonance

$$2k_1 = N, \quad 2k_2 = N, \quad k_1 + k_2 = N, \quad k_1 - k_2 = N \quad (N \in \mathbf{N})$$

the characteristic exponents of the linear system are pure imaginary if $|h - h_*|$ is small enough. We have

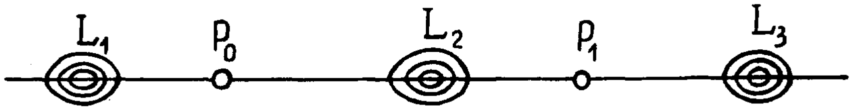

Fig. 4.

$$k_1^2 + k_2^2 = n + 3, \quad 2k_1 k_2 = |n-1|\sqrt{3\nu}$$

Therefore

$$(k_1 \pm k_2)^2 = n + 3 \pm |n-1|\sqrt{3\nu} < 2(n+3)$$

and when $-3 < n < -1$ there are only two possible two-frequency resonances $k_1 \pm k_2 = 1$. This happens when $\nu = 1/3(n+2)^2/(n-1)^2$, with the plus sign when $-3 < n < -2$ and the minus sign for $-2 < n < -1$. Next

$$(n+3)/2 < k_1^2 < (n+3), \quad 0 < k_2^2 < k_1^2$$

and the following single-frequency resonances are possible in the range of n values under consideration

$$2k_1 = 1 \; (-2.75 < n \leqslant -2.5), \quad 2k_2 = 1 \; (-2.5 \leqslant n < -1) \quad \text{for} \quad \nu = \frac{4n+11}{12(n-1)^2}$$

$$k_1 = 1 \; (-2 < n - 1) \quad \text{for} \quad \nu = \frac{4}{3}\frac{n+2}{(n-1)^2}$$

Consequently, the results obtained in [26] imply the following theorem.

Theorem 3. Periodic triangular motions of the unrestricted three-body problem sufficiently close to constant motions are stable in the linear approximation if

$$\nu < \frac{1}{3}\left(\frac{n+3}{n-1}\right)^2; \quad \nu \neq \frac{1}{3}\left(\frac{n+2}{n-1}\right)^2, \quad \frac{4n+11}{12(n-1)^2}, \quad \frac{4}{3}\frac{(n+2)}{(n-1)^2} \quad (-3 < n < -1)$$

Remark. The result formulated in this theorem is due to Lyapunov [3]. It is proved here as a corollary of the fact that system (5.5) is reversible and refined, in the sense that resonance values of ν are eliminated.

We will now consider the case of arbitrary eccentricities. For any $n > -3$, Eq. (5.1) has only periodic solutions. This means, in particular, that when $n > -3$ Eq. (5.2) has pure imaginary characteristic exponents. Consequently, when $n > -3$ the problem of the stability of triangular solutions (circular and elliptical) is solved in the linear approximation by system (5.5), which describes the variation in the configuration of the triangle $P_0P_1P_2$. On the other hand (Section 4), the variation of the configuration may be investigated within the framework of the restricted three-body problem.

We have thus established the following theorem.

Theorem 4. When $n > -3$ the problem of the stability in the linear approximation of triangular solutions of the three-body problem in the general (unrestricted) formulation is equivalent to the same problem in the restricted formulation.

System (5.5) settles the question of stability both in the unrestricted problem and in the special case that we know as the restricted problem. In all cases, the essential parameter is ψ, which is defined in the restricted problem as $\nu = \mu(1-\mu)$, where $\mu = m_1$. In the case of Newtonian interaction ($n = -2$) the domain of stability for the restricted three-body problem was determined by Danby in [27], in the (μ, e) plane. Thus, the domain of stability for the general case is derived from Danby's domain, but now in the (ν, e) plane. This domain is shown in Fig. 5. The resonance curves constructed in Danby's domain [24] are also preserved.

Theorem 5. For $n = -2$ and arbitrary eccentricities $0 \leqslant e < 1$ triangular Laplace solutions of the general (unrestricted) three-body problem are stable everywhere in Danby's domain of stability mapped into the (ν, e) plane, where $\nu = \mu(1-\mu)$, and μ is the relative mass in the restricted three-body problem.

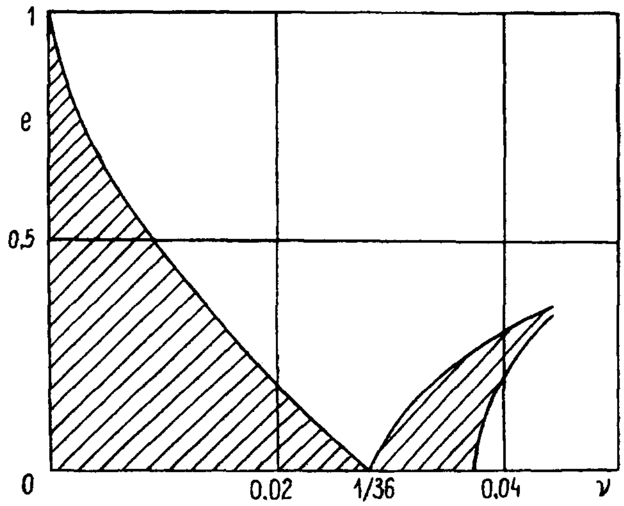

Fig. 5.

6. THE INFLUENCE OF THE BODY P_2 ON THE MOTION OF P_0 AND P_1. RESONANCE INSTABILITY

It follows from the equations of motion (4.1) and (4.2) that, if the relative mass m_2 of $\mathbf{P}_2$ is small, the influence of $\mathbf{P}_2$ on the motion of $\mathbf{P}_0$ and $\mathbf{P}_1$ will be weak. In general, the bodies $\mathbf{P}_0$ and $\mathbf{P}_1$, in turn, will strongly affect the body $\mathbf{P}_2$. However, this influence is limited by the coefficient of r^{n+3} in the formula for the force, and for small changes in r the changes in the solutions of the equations for ψ and y would appear to be small.

Let us consider two cases in which the size of the triangle and its configuration affect one another strongly. In the first case the originally circular orbit of the bodies $\mathbf{P}_0$ and $\mathbf{P}_1$ is replaced by a neighbouring elliptical orbit, and the change turns out to be "fatal" for the motion of $\mathbf{P}_2$ in the neighbourhood of a triangular libration point. In the second case the interaction is non-linear, and not only do $\mathbf{P}_0$ and $\mathbf{P}_1$ "strongly" affect $\mathbf{P}_2$, but the converse is also true: $\mathbf{P}_2$ "strongly" affects the motion of $\mathbf{P}_0$ and $\mathbf{P}_1$. Both cases belong to the resonance category. With an eye on the qualitative picture only, we shall confine our attention to the case of Newtonian interaction (n = 2).

Let $\nu = 1/36$. One then has a parametric resonance $2k_2 = 1$ in the elliptical problem. At the same time, as follows from (5.2), we have a frequency $k_0 = \sqrt{(n + 3)}$ in the circular problem, and when $n = -2$ one obtains an internal resonance $k_0 = 2k_2$ [28, 29]. As can be seen from system (4.3), the resonance $k_0 = 2k_2$ does not cause resonance instability when $m_2 = 0$, but at resonance $2k_2 = 1$ the triangular libration points of the restricted three-body problem are unstable [24].

1. Let us consider the stability of the Lagrangian solutions at parametric resonance $2k_2 = 1$. When $n = -2$ Eqs (5.4) are the equations of Keplerian motion for a point of unit mass attracted to the origin by a point of mass M. In motion about an ellipse with semi-major axis a we have

$$\rho = \frac{a(1-e^2)}{1+e\cos\theta}, \quad u = \frac{3}{1+e\cos\theta}$$

(e is the eccentricity). Therefore, our problem reduces to investigating the linear periodic system

$$X'' - 2Y' = (1-\lambda)uX, \quad Y'' + 2X' = \lambda u Y \quad (\lambda^2 - \lambda + 1/48 = 0) \tag{6.1}$$

for small e.

Let us expand $u(\theta)$ in Fourier series

$$u = 3(1-e^2)^{-1/2}[1 - e_1\cos\theta + e_1^2\cos 2\theta + \dots], \quad e_1 = e/(1+\sqrt{1-e^2})$$

The roots of the characteristic equation of the averaged system will therefore be

$$\varkappa_{1,2}^2(e) = \frac{-(4-\bar{u}) \pm \sqrt{(4-\bar{u})^2 - 3\nu\bar{u}^2}}{2}, \quad \bar{u} = 3/\sqrt{1-e^2}$$

and the numbers $\varkappa(e)$ are pure imaginary when

$$\nu \leqslant (4-\bar{u})^2/(3\bar{u}^2) \tag{6.2}$$

Consequently, if this condition is satisfied, the Lagrangian periodic solutions are stable [26] in the linear approximation, provided there are no second-order resonances. Note that in (6.2) it is surely true that $\nu \leqslant 1/27$.

For small e there is a single second-order resonance in the domain (6.2), namely, $2\varkappa_2 = i$. In that case

$$\nu = (5\sqrt{1-e^2} - 4)\sqrt{1-e^2}/36$$

and system (6.1) becomes

$$z_1' = \varkappa_1(e)z_1 + \ldots, \quad z_2' = \varkappa_2(e)z_2 + eR(e)\exp(i\theta)\bar{z}_2 + \ldots$$

where z and $\bar{z}$ are complex-conjugate variables. If $R(e) \neq 0$, parametric resonance leads to instability.

To determine the resonance coefficient $R(e)$, we change the variables

$$\begin{aligned}
z_1 &= \alpha\bar{u}[(2i+\varkappa_1)p + p' + (\varkappa_1 - 2i)q] + (\varkappa_1^2 - 2i\varkappa_1 - \bar{u}/2)q' \\
z_2 &= \alpha\bar{u}[(2i+\varkappa_2)p + p' + (\varkappa_2 - 2i)q] + (\varkappa_2^2 - 2i\varkappa_2 - \bar{u}/2)q' \\
p &= X + iY, \quad q = X - iY, \quad \alpha = 1/2 - \lambda
\end{aligned}$$

Then Eqs (6.1) become

$$z_j' = \varkappa_j z_j + \left[\alpha\bar{u}\left(\frac{p}{2} + \alpha q\right) + \left(\varkappa_j^2 - 2i\varkappa_j - \frac{\bar{u}}{2}\right)\left(\frac{q}{2} + \alpha p\right)\right](u - \bar{u}), \quad j = 1,2 \tag{6.3}$$

For p and q on the right of this equation one must substitute the expressions

$$\begin{aligned}
p &= i\alpha\bar{u}(z_1 + z_2) - i\left(\varkappa_1^2 + 2i\varkappa_1 - \frac{\bar{u}}{2}\right)\bar{z}_1 - i\left(\varkappa_2^2 + 2i\varkappa_2 - \frac{\bar{u}}{2}\right)\bar{z}_2 \\
q &= -i\alpha\bar{u}(\bar{z}_1 + \bar{z}_2) + i\left(\varkappa_1^2 - 2i\varkappa_1 - \frac{\bar{u}}{2}\right)z_1 + i\left(\varkappa_2^2 - 2i\varkappa_2 - \frac{\bar{u}}{2}\right)z_2
\end{aligned} \tag{6.4}$$

In view of formulae (6.3) and (6.4), the formula for $R(e)$ is

$$R(e) = \frac{3\alpha\bar{u}i}{2(1+\sqrt{1-e^2})}\left(\varkappa_2^2 + \alpha^2\bar{u} - \frac{1+3\nu}{4}\bar{u}\right)$$

Noting that

$$\varkappa_2 = i/2, \quad \alpha^2 = 11/48, \quad \bar{u} = 3 + \ldots, \quad \nu = 1/36 + \ldots$$

we obtain $R(e) \neq 0$, and moreover $R(0) \neq 0$.

Consequently, the resonance $2\varkappa_2 = i$. induces instability. This conclusion follows from an analysis of the terms linear in e. Therefore, if the resonance condition is satisfied up to terms of order e^2, all our previous conclusions remain valid. Under these conditions the characteristic exponents of system (6.1) are $\pm eR(0) \pm i/2 + O(e^2)$. A domain of instability appears in the (ν, e) plane in the neighbourhood of the point (1/36, 0). This is a well-known result for the restricted problem [24].

Theorem 6. An elliptical Lagrangian solution close to a circular solution is unstable at $\nu = 1/36$.

2. Let us consider the stability of constant Lagrangian solutions at resonance $k_0 = 2k_2$. As follows from [30, 31], in this case resonance instability is derived from the following system, normalized up to terms of second order inclusive and expressed in terms of complex-conjugate variables z and $\bar{z}$

$$\dot{z_0} = iz_0 + iB_0\bar{z}_2^2 + \dots$$

$$\dot{z_1} = ik_1z_1 + \dots, \quad \dot{z_2} = -iz_2/2 + iB_2\bar{z}_0\bar{z}_2 + \dots$$

where B_0 and B_2 are real constants. This system is obtained by a linear transformation of the variational equations to normal coordinates, after which one equates all the coefficients of second-order terms, except the resonance coefficients, to zero [32]. In addition, the variables $z_0, \bar{z}_0$ correspond to the equation for r and the variables $z_1, \bar{z}_1, z_2, \bar{z}_2$ correspond to the equations for ψ and y. It is now clear from the form of the resonance terms that the term $iB_0\bar{z}_2^2$ may be obtained only from rF_2 in (4.2), while the term $iB_2\bar{z}_0\bar{z}_2$ is determined only by terms involving r^{n+3} in (4.1).

The above considerations considerably simplify the computation of the coefficients B_0 and B_2. Thus, B_2 already appears in the written-out part of system (5.3) if one puts $u = 3(1 + \xi)$; in unperturbed motion, $r = \sqrt{(\nu)}\rho, fM\rho = \beta_*^2$.

We will first use (5.2) to get an equation for z_0 from (4.2). We have

$$\xi'' + \xi = \frac{\mu}{S^2}(\psi'^2 + y'^2) + \dots$$

where we have actually written out only those quadratic terms that make a contribution to B_0. In variables

$$z_0 = \xi - i\xi', \quad \bar{z}_0 = \xi + i\xi'$$

we then obtain

$$z_0' = iz_0 - \frac{\mu i}{S^2}(\psi'^2 + y'^2) + \dots$$

Now, applying the linear transformation

$$z_j = -\frac{2(1-\lambda)u}{\lambda u - \lambda_j^2}X + \frac{\lambda u}{\lambda_j}Y - \frac{2\lambda_1}{\lambda u - \lambda_j^2}X' + Y'$$
$$\bar{z}_j = -\frac{2(1-\lambda)u}{\lambda u - \lambda_j^2}X - \frac{\lambda u}{\lambda_j}Y + \frac{2\lambda_1}{\lambda u - \lambda_j^2}X' + Y', \quad j = 1,2 \tag{6.5}$$

where $\lambda_1 = -i\sqrt{(3)}/2$, $\lambda_2 = -i/2$ are the roots of the characteristic equation, $u = 3$, we transform the linear part of system (5.5) to normal coordinates. Then system (5.5) takes the form

$$z_j' = \lambda_j z_j - \frac{2\lambda_j}{\lambda u - \lambda_j^2}(1-\lambda)u\xi X + \lambda u\xi Y + \dots, \quad j = 1,2 \tag{6.6}$$

Let us find the inverse transformation

$$X = \frac{(\lambda u - \lambda_1^2)(\lambda u - \lambda_2^2)}{4(1-\lambda)u(\lambda_2^2 - \lambda_1^2)}[(z_1 + \bar{z}_1) - (z_2 + \bar{z}_2)]$$

$$Y = \frac{\lambda_1^2\lambda_2^2}{2\lambda^2u^2(\lambda_2^2-\lambda_1^2)}\left[\frac{\lambda u-\lambda_1^2}{\lambda_1}(z_1-\bar z_1)-\frac{\lambda u-\lambda_2^2}{\lambda_2}(z_2-\bar z_2)\right] \tag{6.7}$$

$$X' = \frac{(\lambda u-\lambda_1^2)(\lambda u-\lambda_2^2)}{4\lambda u(\lambda_2^2-\lambda_1^2)}[\lambda_1(z_1-\bar z_1)-\lambda_2(z_2-\bar z_2)]$$

$$Y' = \frac{1}{2(\lambda_2^2-\lambda_1^2)}[(\lambda u-\lambda_1^2)(z_1+\bar z_1)-(\lambda u-\lambda_2^2)(z_2+\bar z_2)]$$

Now, using the relations

$$\psi'^2+y'^2=x'^2+y'^2=(X'^2+Y'^2)/(1+s^2)$$

we calculate the coefficient B_0

$$B_0=-\frac{\mu(\lambda u-\lambda_2^2)}{16\nu^2(1+s^2)\lambda^2u^2(\lambda_2^2-\lambda_1^2)}[(\lambda u-\lambda_1^2)^2\lambda_2^2+4\lambda^2u^2]$$

Next, replacing ξ on the right of (6.6) by the expression $(z_0+\bar z_0)/2$ and replacing X, X', Y, Y' by the right-hand sides of (6.7), we obtain

$$B_2=-\frac{\lambda_2 i}{4\lambda u(\lambda_2^2-\lambda_1^2)}[\lambda^2u^2-\lambda_1^2\lambda_2^2]$$

In the case under consideration

$$u=3,\quad 2\lambda=1\pm\sqrt{11/12},\quad \lambda_1^2=-3/4,\quad \lambda_2^2=-1/4\quad (\lambda_2=-i/2)$$

Therefore $B_0B_2>0$ and we obtain instability.

Theorem 7. Constant Lagrangian solutions are unstable when

$$\nu=m_0m_1+m_0m_2+m_1m_2=1/36$$

Remark. Instability at resonance $k_0=2k_2$ occurs for any $-3<n<-1$ [29]. The analytical proof of this statement follows the same lines as in the case of Newtonian interaction. The transformation formulae (6.5) and (6.7) hold for any n.

7. SYMMETRIC PERIODIC MOTIONS FOR SMALL m_1 AND m_2

Turning now to system (4.1), (4.2), let us investigate the case in which the mass of the body $\mathbf{P}_0$ is significantly greater than those of $\mathbf{P}_1$ and $\mathbf{P}_2$. Letting the parameters m_1 and m_2 tend to zero, we obtain the limiting problem

$$\begin{aligned}&\frac{d}{d\theta}(e^{2y}\psi')=-2e^{2y}y'\\&\frac{d}{d\theta}(e^{2y}y')=2e^{2y}\psi'+e^{2y}(\psi'^2+y'^2)-\frac{fMr^{n+3}}{\beta_*^2}(e^{ny}-e^{y})e^{y}\\&\ddot r-\frac{\beta_*^2}{r^3}+fMr^n=0,\quad r^2\frac{d\theta}{dt}=\beta_*\end{aligned} \tag{7.1}$$

This system has a periodic solution which is symmetric with respect to the fixed set M

$$y=y_0(\text{const}),\quad \psi=\omega\theta,\quad r=r_0(\text{const}),\quad y'=0,\quad \psi'=\omega,\quad r'=0 \tag{7.2}$$

where

$$\omega^2 + 2\omega + 1 - e^{(n-1)y_0} = 0, \quad \beta_*^2 = fMr_0^{n+3} \tag{7.3}$$

In this solution the bodies $\mathbf{P}_1$ and $\mathbf{P}_2$ describe circles about $\mathbf{P}_0$, at respective angular velocities β_*/r_0^2 and $(\omega + 1)\beta_*/r_0^2$. Depending on the sign of $\omega + 1$, motion takes place in one direction or in two opposite directions.

We now consider the problem of continuing the motions (7.2) to small parameters m_1 and m_2. To that end, using the previous results of [33], we set up the variational equations for (7.1)

$$\begin{aligned} &\delta\psi'' = -2\delta y' / \omega \\ &\delta y'' = 2(1+\omega)\delta\psi' / \omega - [e^{(n-1)y_0}(n-1)\delta y - (n+3)\delta\xi] / \omega^2 \\ &\delta\xi'' + (n+3)\delta\xi / \omega^2 = 0, \quad r = r_0(1+\xi) \end{aligned} \tag{7.4}$$

where the prime now denotes differentiation with respect to the variable $\omega\theta$.

The characteristic equation of this system has a pair of zero roots with one group of solutions. These roots can be continued with respect to the parameters [33]. Therefore, the problem of whether the motions can be continued is solved using the remaining roots, which are

$$\varkappa_{1,2} = \pm i\sqrt{n+3} / \omega, \quad \varkappa_{3,4} = \pm\sqrt{(1-n)\exp\{(n-1)y_0\} - 3} / \omega$$

Then [33], when

$$n + 3 \neq N^2\omega^2, \quad 3 + (n-1)\exp\{(n-1)y_0\} \neq N^2\omega^2 \quad (N \in \mathbf{N})$$

symmetric periodic solutions near (7.2), (7.3) exist for sufficiently small $m_1 \neq 0, m_2 \neq 0$.

We might mention that the question of whether other types of symmetric periodic orbits exist has been considered for Newtonian interaction in [34] within the framework of the many-body problem.

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